Lecture 9

Friday, May 13, 2022 12:46 PM

* Prayer
* Spiritual thought

Chain rule:

Calc I: $\{(x+1)^4\}' = 4(x+1)'(x+1)^3 = 4(x+1)^3$

$f = f(x,y)$, $x = x(u,v)$, $y = y(u,v)$

$f_u = ?$, $f_v = ?$

Ex $z = x^4 - y^4$

$x = uv$, $y = u+v$

$z = (uv)^4 - (u+v)^4$

$dz = z_x\,dx + z_y\,dy$

$dx = x_u\,du + x_v\,dv$

$dy = y_u\,du + y_v\,dv$

$$dz = \underbrace{(z_x x_u + z_y y_u)}_{z_u}du + \underbrace{(z_x x_v + z_y y_v)}_{z_v}dv$$

Multivariable chain rule:

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}}$$

Ex

$f(x,y)$ satisfies $f_{xx}+f_{yy}=0$

$$\begin{cases} x = u-2v \\ y = 2u+v \end{cases}$$

$g(u,v) = f(u-2v,\ 2u+v)$

Show that $g_{uu}+g_{vv}=0$.

Ex: $x^2 + y\sin z + zx = 1$ $(*)$

Find z_x and z_y at $(1,1,0)$.

We view z as a function of x and y.

Differentiate both sides of $(*)$ wrt x. ← This is called implicit differentiation

$$2x + 0 + z_x x + z = 0$$

$$\rightsquigarrow \quad z_x = \frac{-2x-z}{x}$$

At $(x,y,z) = (1,1,0)$, $z_x = \dfrac{-2-0}{1} = -2$

Differentiate both sides of $(*)$ wrt y:

$$\sin z + y z_y \sin z + z_y x = 0$$

$$\rightsquigarrow \quad z_y = \frac{-\sin z}{y\sin z + x}$$

At $(x,y,z) = (1,1,0)$, $z_y = \dfrac{-0}{0+1} = 0$.

Ex

$$xy^2 + \sin y = 0 \qquad (*)$$

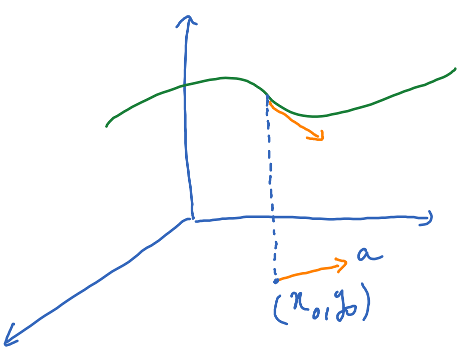
Find y_x at $(x,y) = (0, \pi)$.

Diff. $(*)$ wrt x: $\quad y^2 + 2x y_x + y_x \cos y = 0$

$$\rightsquigarrow y_x = -\frac{y^2}{2x + \cos y}$$

At $(x,y) = (0, \pi)$, $\quad y_x = -\frac{\pi^2}{\cos \pi} = \pi^2.$

Directional derivative



$$D_a f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha_1, y_0 + ha_2) - f(x_0, y_0)}{h}$$

$a = (a_1, a_2)$ is a unit vector

Thm:

$$D_a f(x_0, y_0) = \nabla f(x_0, y_0) \cdot a$$

Ex: $\quad f(x,y) = x^2 + xy + y^2$

Find directional derivative at $P(1,2)$ in the direction of $\overrightarrow{PQ}$ with $Q(0,1)$.

$$\overrightarrow{PQ} = (-1, -1) \longrightarrow a = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$\nabla f = (2x + y, 2y + x)$$

$\nabla f(1,2) = (4,5)$

$D_a f(1,2) = \nabla f(1,2) \cdot a = (4,5) \cdot \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{9}{\sqrt{2}}$.

Consequence: gradient is the direction where f increases the most

$D_a f(x_0, y_0) \to \max$

$\nabla f(x_0, y_0) \cdot a \to \max$: happens where $a = \frac{\nabla f}{|\nabla f|}$

Level set is where f doesn't change its value.

Observation:

gradient is always perpendicular to level set.

Use Mathematica to draw level sets and gradient field.

Application:

Ellipsoid $x^2 + 2y^2 + 2z^2 = 2$.

Find tangent plane at $\left(1, -\frac{1}{2}, \frac{1}{2}\right)$.

$g(x,y,z) = x^2 + 2y^2 + 2z^2$

The ellipsoid is a 2-level set of g. This level set is perpendicular to the gradient of g at $\left(1, -\frac{1}{2}, \frac{1}{2}\right)$.

$\nabla g = (2x, 4y, 4z)$

$\nabla g(1, -\frac{1}{2}, \frac{1}{2}) = (2, -2, 2)$ $\leftarrow$ This is a normal vector to the tangent plane.

The tangent plane has an equation

$$2(x-1) - 2(y+\tfrac{1}{2}) + 2(z-\tfrac{1}{2}) = 0$$

$\rightsquigarrow$ $2x - 2y + 2z - 4 = 0$

$\rightsquigarrow$ $x - y + z - 1 = 0$.